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Recovering Elastic Moduli of Nonlinear Anisotropic Annular Membranes from Interior Measurements of Deformation and Pressure

Arthur David Cummings
Second Lieutenant, United States Air Force

1995

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We investigate the problem of inferring elastic moduli of nonlinearly elastic membranes from interior measurements of deformation and pressure. We begin by formulating a model of membrane deformation under a vertical force where the geometry of the membrane is star-like. The model makes no specification of the constitutive law by which stresses are calculated from applied strains. Under appropriate choice of the constitutive law and simplification of the geometry, we show that membranes of regular structure may be homogenized to an axisymmetric case. We then investigate numerical methods for the resolution of the axisymmetric model in terms of radial and vertical displacement. Examples are given for various boundary conditions and choices for elastic moduli. We present a method by which the moduli may be accurately recovered by algebraic calculation from knowledge of the displacements on the boundary and interior of the membrane together with measurement of the radial stress at one of the boundaries.

RICE UNIVERSITY

Recovering Elastic Moduli of Nonlinear Anisotropic Annular Membranes from Interior Measurements of Deformation and Pressure

by

Arthur David Cummings

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Master of Arts

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We investigate the problem of inferring elastic moduli of nonlinearly elastic membranes from interior measurements of deformation and pressure. We begin by formulating a model of membrane deformation under a vertical force where the geometry of the membrane is star-like. The model makes no specification of the constitutive law by which stresses are calculated from applied strains. Under appropriate choice of the constitutive law and simplification of the geometry, we show that membranes of regular structure may be homogenized to an axisymmetric case. We then investigate numerical methods for the resolution of the axisymmetric model in terms of radial and vertical displacement. Examples are given for various boundary conditions and choices for elastic moduli. We present a method by which the moduli may be accurately recovered by algebraic calculation from knowledge of the displacements on the boundary and interior of the membrane together with measurement of the radial stress at one of the boundaries.

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Chapter 1

Introduction

The task of parameter identification from some measurement of physical phenomena has application in various fields. Across these fields, there seem to be two major approaches in measuring the physical occurrences. Both approaches can be described in terms of the following problem:

$$-\nabla \cdot (a \nabla u) = f \text{ in } \Omega \tag{1.1}$$

$$u = g \text{ on } \partial\Omega \tag{1.2}$$

where Ω is a region in $\Re^2$. The function a is the parameter that we want to identify, while u is the measurable phenomenon.

The first approach seeks to gain information about a over the entire domain by measuring u only on the boundary of the object. In this case, the function f would be known over the entire domain, and g would be known over the entire boundary. Such an approach has appeal in that it is not intrusive to the interior of the object. In application, such an intrusion may disturb the normal functioning of the object, or it may be infeasible for other reasons to observe phenomena on the interior of the object. Nakamura and Uhlmann [17] discuss identification of the Lamé parameters which determine the elastic properties of a linear, non-homogeneous, isotropic elastic medium from boundary measurements. The success of their method relies on the determination of a mapping Λ from the Dirichlet boundary conditions of (1.2) to Neumann boundary conditions of the form

$$a\frac{\partial u}{\partial \overline{n}} = \Lambda g \tag{1.3}$$

where $\overline{n}$ is the outward unit normal to $\partial\Omega$, and a depends on the Lamé parameters. Their discussion is representative of this first approach to parameter identification.

A second approach involves measurement of the phenomena u on the interior of the body. As in the first approach, f and g are known. However, here u is known on the interior of Ω as well. Hence a is the only quantity which is unknown. An example of

this approach is provided by Kohn and Lowe [16] who present a variational method for parameter identification for a similar problem in water flow. They also review other methods of parameter estimation which use interior measurements as part of their approach.

In this thesis, we will concern ourselves with inferring the elastic moduli of a nonlinearly elastic membrane. Our interest in this application arises from an effort to better understand the mechanics of the diaphragm muscle. It is intended that this work will be integrated with ongoing research in *in vivo* animal physiologic experimentation taking place in the Pulmonary Section of the Baylor College of Medicine. This experimentation involves measurement of the deformation of the diaphragm under a known pressure. Knowledge of this deformation is analogous to knowing u on both the interior and the boundary of Ω in our example problem. Knowledge of the pressure is similar to knowing f. Boriek, Wilson, and Rodarte [2] present the details of how these measurements are made. Since their methodology does yield interior measurements of deformation, we will focus on the second approach to parameter identification.

An important preparation for recovering the elastic moduli is to determine the form of the constitutive law that predicts the state of stress from knowledge of strain. In the case of the diaphragm, this can be extremely difficult. Figure 1.1 shows a picture of a canine diaphragm. From this photograph, it is clear that the diaphragm has a complicated structure. Near the chest wall, we can see that radial bands of collagen alternate with muscle bundles as we move around the central tendon. Although the diaphragm is not homogeneous, from this viewpoint it does seem to have some regular structure. In this thesis, we will develop a model which will allow us to consider some aspects of this structure. We also present a method by which we are able to recover the elastic moduli of nonlinear axisymmetric membranes from the interior measurements of the deformation.

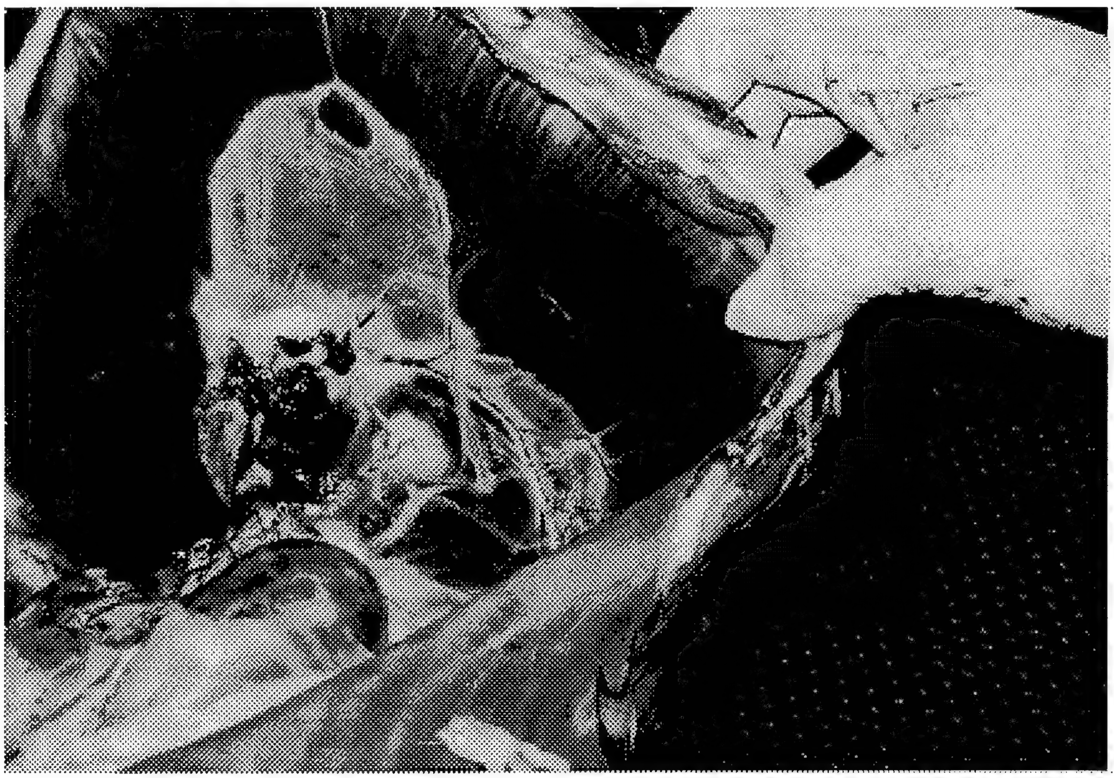

Figure 1.1 A picture of the canine diaphragm. Notice the alternating structure of the lighter collagen bands with the darker muscle bundles near around the central tendon.

Chapter 2

Models of Membrane Deformation

2.1 General Membrane Models

The first step in recovering the elastic moduli which describe the material properties of the diaphragm muscle is to develop an appropriate mathematical model which describes the deformation of the muscle when stress is applied. We have chosen to model the diaphragm by an engineering structure known as a membrane. Since we do not wish to include in our model the strong central tendon of the diaphragm, we can represent the diaphragm by a star-like geometry with a star-like hole. We then would choose boundary conditions at the interior boundary in attempt to mimic the interface between the diaphragm and the central tendon. Further simplification to a regular annulus would allow us to consider models of axisymmetric membranes.

The concept of a membrane is a simplification of a more general three-dimensional engineering structure known as a shell. A shell may be subjected to not only stresses, but moments as well. As a result, the equations for the bending of shells can be quite difficult to formulate. Green and Adkins [9] note that if the changes in dimension vary only slightly throughout the thickness of the deformed shell, then shearing stresses in this direction and couples are then negligible. This is the motivation for the theory of elastic membranes. Thus membrane theory neglects shears normal to the plane of the membrane, as well as both bending and twisting moments. These assumptions are reasonable for the passive diaphragm which we are trying to model. They would not hold for an active diaphragm.

Green and Adkins formulate the theory of membranes independently of the theory of shells. Antman [1], however, first derives equations for shells, then applies the simplifying assumptions which bring us to membrane theory. In both cases, a great deal of attention is given to the axisymmetric membrane. Antman's approach provides a great deal of flexibility in specifying the undeformed geometry of the axisymmetric membrane, as well as versatility in specifying the form of the body force to which the membrane is subjected. This allows him to consider not only plate deformations, but

the deformation of spherical membranes. Dickey [6] presents in his paper a model for the deformation of an initially flat, hyperelastic circular membrane under vertical pressure. His equations are an instance of the more general equations derived by Antman.

In section 2.2 we develop a general model for deformation of a non-homogeneous star-like membrane subjected to a vertical pressure by generalizing the derivation of Dickey's model to this domain. We do this by first determining the strains associated with the deformation. From the resulting balance of forces in the deformed state, we then derive the equations of nonlinear elasticity. This derivation places no restriction on the form of the stress-strain relationship. We therefore consider some relevant forms of the constitutive law. In particular, we discuss a stress-strain relationship which allows periodic structures such as that of the diaphragm shown in Figure 1.1 to be considered macroscopically homogeneous. In section 2.3 we apply this constitutive law to homogenize the stiffness of this membrane. With additional geometric simplifications, we then arrive at an axisymmetric form of the equations of nonlinear elasticity.

2.2 A Model of a Star-like Membrane Under a Vertical Force

In this section, we will develop a model for a membrane which can be expressed in polar coordinates, but is not axisymmetric. Therefore, the model must allow for the membrane to vary with θ. In doing so, we will generalize the approach which Dickey [6] sets forth in the derivation of his axisymmetric model.

We define the reference state of the membrane of thickness h to be, in Cartesian coordinates,

$$(r\cos\theta,\ r\sin\theta,\ z(r,\theta))$$

for r and θ belonging to the domain

$$\Omega = \{(r,\theta) : f_1(\theta) < r < f_2(\theta), \theta \in [0, 2\pi)\}$$

where f_1 and f_2 are continuous 2π periodic functions of θ. Note that the reference state need not be flat. The vertical reference configuration need only be representable as a differentiable function of r and θ. Due to the application of a vertical pressure of magnitude P, the membrane is deformed to the point

$$((r+u(r,\theta))\cos(\theta+\phi(r,\theta)),\ (r+u(r,\theta))\sin(\theta+\phi(r,\theta)),\ z(r,\theta)+w(r,\theta))$$

where u, ϕ, and w indicate radial, angular, and vertical displacements, respectively. We specify the following displacements as boundary conditions:

$$\begin{gathered} u(f_1(\theta),\theta) = \alpha_u(\theta), \quad u(f_2(\theta),\theta) = \beta_u(\theta), \\ w(f_1(\theta),\theta) = \alpha_w(\theta), \quad w(f_2(\theta),\theta) = \beta_w(\theta), \\ \phi(r,0) = \phi(r,2\pi). \end{gathered} \tag{2.1}$$

We should note that we may specify conditions on the radial stress at the inner and outer radius instead of the radial displacement. Introducing the function σ_r representing radial stress, we can then state the boundary conditions as

$$\begin{gathered} \sigma_r(f_1(\theta),\theta) = \alpha_\sigma(\theta), \quad \sigma_r(f_2(\theta),\theta) = \beta_\sigma(\theta), \\ w(f_1(\theta),\theta) = \alpha_w(\theta), \quad w(f_2(\theta),\theta) = \beta_w(\theta), \\ \phi(r,0) = \phi(r,2\pi). \end{gathered} \tag{2.2}$$

To simplify notation, we will not generally include the arguments of the functions u, w, z, and ϕ, but will assume them to be understood. We will indicate the partial derivative of one of these functions with respect to one of the arguments by the function subscripted by a comma and the argument (i.e. $u_{,r} = \frac{\partial u}{\partial r}$). The subscripts r or θ without a comma indicate direction, and are not to be interpreted as differentiation.

2.2.1 Derivation of the Model

Derivation of Strain Definitions

The first step in the derivation of the model is to determine the equations which define strain in the radial and circumferential directions. These equations are derived from differential considerations of the deformation function in the appropriate direction using the basic definition of strain:

$$\text{strain} = \frac{(\text{deformed length}) - (\text{undeformed length})}{\text{undeformed length}}.$$

Tensile Strain in the Circumferential Direction First, let us consider the deformation of an arc of length $\sqrt{r^2 + z_{,\theta}^2}\, d\theta$ for a given value of r. Note that for a given r, the relationship above describes a set of functions which are parameterized by θ where

$$\begin{aligned} x^* &= [r + u(r,\theta)]\cos(\theta + \phi(r,\theta)) \\ y^* &= [r + u(r,\theta)]\sin(\theta + \phi(r,\theta)) \\ z^* &= z(r,\theta) + w(r,\theta). \end{aligned}$$

Thus the length of the deformed arc is ds^* where

$$(ds_\theta^*)^2 = \left(\frac{dx^*}{d\theta}\right)^2 + \left(\frac{dy^*}{d\theta}\right)^2 + \left(\frac{dz^*}{d\theta}\right)^2.$$

Now, we have

$$\frac{dx^*}{d\theta} = u_{,\theta}\cos(\theta+\phi) - (r+u)(1+\phi_{,\theta})\sin(\theta+\phi) \tag{2.3}$$

$$\frac{dy^*}{d\theta} = u_{,\theta}\sin(\theta+\phi) + (r+u)(1+\phi_{,\theta})\cos(\theta+\phi) \tag{2.4}$$

$$\frac{dz^*}{d\theta} = z_{,\theta} + w_{,\theta} \tag{2.5}$$

Thus,

$$(ds_\theta^*)^2 = u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2.$$

Therefore, the initial differential arc length of $r\,d\theta$ is deformed to

$$\sqrt{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}\,d\theta.$$

Thus we can define the tensile strain in the circumferential direction as

$$\begin{aligned}\mathcal{E}_\theta &= \frac{\sqrt{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}\,d\theta - \sqrt{r^2+z_{,\theta}^2}\,d\theta}{\sqrt{r^2+z_{,\theta}^2}\,d\theta} \\ &= \sqrt{\frac{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}{r^2+z_{,\theta}^2}} - 1 \qquad (2.6)\end{aligned}$$

Tensile Strain in the Radial Direction Similarly, let us consider the curve of length $\sqrt{1+z_{,r}^2}\,dr$, holding θ to be constant. We know that the deformed arc length is defined similarly by

$$(ds_r^*)^2 = \left(\frac{dx^*}{dr}\right)^2 + \left(\frac{dy^*}{dr}\right)^2 + \left(\frac{dz^*}{dr}\right)^2.$$

Now, we have that

$$\frac{dx^*}{dr} = (1+u_{,r})\cos(\theta+\phi) - (r+u)\phi_{,r}\sin(\theta+\phi) \tag{2.7}$$

$$\frac{dy^*}{dr} = (1+u_{,r})\sin(\theta+\phi) + (r+u)\phi_{,r}\cos(\theta+\phi) \tag{2.8}$$

$$\frac{dz^*}{dr} = z_{,r} + w_{,r} \tag{2.9}$$

Thus

$$(ds_r^*)^2 = (1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2.$$

We also define the tensile strain in the radial direction as

$$\begin{aligned}
\mathcal{E}_r &= \frac{\sqrt{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}\, dr - \sqrt{1+z_{,r}^2}\, dr}{\sqrt{1+z_{,r}^2}\, dr} \\
&= \sqrt{\frac{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}{1+z_{,r}^2}} - 1 \qquad (2.10)
\end{aligned}$$

Shear Strain We now introduce an appropriate set of orthogonal unit vectors for use in describing the deformed state. These are

$$\overline{r} = \cos(\theta+\phi)\,\overline{\imath} + \sin(\theta+\phi)\,\overline{\jmath} \qquad (2.11)$$

$$\overline{\theta} = -\sin(\theta+\phi)\,\overline{\imath} + \cos(\theta+\phi)\,\overline{\jmath} \qquad (2.12)$$

where $\overline{\imath}, \overline{\jmath}$, and $\overline{k}$ are the usual unit vectors in the x, y, and z directions. Note that $\overline{r}$ and $\overline{\theta}$ are functions of r and θ.

With this basis for the xy-plane defined, we now are interested in how the deformed segments described in sections 2.2.1 and 2.2.1 are oriented with respect to these basis vectors. We will consider only the projections of these segments into the xy-plane.

Shear Strain in the Circumferential Direction The slope of the segment in section 2.2.1 projected into the xy-plane is

$$\begin{aligned}
\frac{dy^*}{dx^*} &= \frac{dy^*/d\theta}{dx^*/d\theta} \\
&= \frac{u_{,\theta}\sin(\theta+\phi) + (r+u)(1+\phi_{,\theta})\cos(\theta+\phi)}{u_{,\theta}\cos(\theta+\phi) - (r+u)(1+\phi_{,\theta})\sin(\theta+\phi)}
\end{aligned}$$

The slope of $\overline{\theta}$ is

$$-\frac{\cos(\theta+\phi)}{\sin(\theta+\phi)} = -\cot(\theta+\phi).$$

We will call the angle between these two slopes γ_1, and will associate a positive angle with a deflection in the clockwise direction. Thus, by use of a trigonometric identity, we find that

$$\begin{aligned}
\tan\gamma_1 &= \frac{-\cot(\theta+\phi) - \frac{dy^*}{dx^*}}{1 - \cot(\theta+\phi)\frac{dy^*}{dx^*}} \\
&= \frac{u_{,\theta}}{(r+u)(1+\phi_{,\theta})} \qquad (2.13)
\end{aligned}$$

Shear Strain in the Radial Direction The slope of the segment in section 2.2.1 projected into the xy-plane is

$$\begin{aligned}\frac{dy^*}{dx^*} &= \frac{dy^*/dr}{dx^*/dr}\\ &= \frac{(1+u_{,r})\sin(\theta+\phi)+(r+u)\phi_{,r}\cos(\theta+\phi)}{(1+u_{,r})\cos(\theta+\phi)-(r+u)\phi_{,r}\sin(\theta+\phi)}\end{aligned}$$

The slope of $\overline{r}$ is

$$\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \tan(\theta+\phi).$$

We will call the angle between these two slopes γ_2, and will associate a positive angle with deflection in the counter-clockwise direction. Thus, by use of the same trigonometric identity, we find that

$$\begin{aligned}\tan\gamma_2 &= \frac{\frac{dy^*}{dx^*}-\tan(\theta+\phi)}{1+\tan(\theta+\phi)\frac{dy^*}{dx^*}}\\ &= \frac{(r+u)\phi_{,r}}{1+u_{,r}} \qquad (2.14)\end{aligned}$$

Total Shear Strain The total shear strain is the sum of γ_1 and γ_2. Thus we may define the shear strain as

$$\gamma = \arctan\left(\frac{u_{,\theta}}{(r+u)(1+\phi_{,\theta})}\right) + \arctan\left(\frac{(r+u)\phi_{,r}}{1+u_{,r}}\right).$$

Applying the same trigonometric identity, we may rewrite the above equation as

$$\tan\gamma = \frac{u_{,\theta}(1+u_{,r})+(r+u)^2(1+\phi_{,\theta})\phi_{,r}}{(r+u)[(1+\phi_{,\theta})(1+u_{,r})-u_{,\theta}\phi_{,r}]} \qquad (2.15)$$

Equilibrium Conditions and Resulting Equations

The deformed state of the membrane occurs when all forces are in equilibrium, i.e., when the membrane stresses balance the applied vertical pressure. Having determined the tensile and shear strains, we are able to express both the orientation and the area of the faces on which the forces act. This will enable us to determine the equations that must be satisfied in order for equilibrium to occur.

Unit Normal and Tangent Vectors to the Deformed Faces In order to consider a balance of forces, we must first understand the directions in which the forces act. Also we find it useful to express these vectors in terms of the unit vectors $\overline{r}$ and $\overline{\theta}$.

Unit Tangent Vectors From equations (2.3)–(2.5), we can obtain a convenient representation for a vector tangent to the face $r = constant$ as follows.

$$\bar{t}_1 = \frac{dx^*}{d\theta}\,\bar{\imath} + \frac{dy^*}{d\theta}\,\bar{\jmath} + \frac{dz^*}{d\theta}\,\overline{k}$$

We find the components of $\bar{t}_1$ in terms of $\bar{r}$, $\overline{\theta}$, and $\overline{k}$ to be

$$\begin{aligned} \bar{t}_1 \cdot \bar{r} &= u_{,\theta} \\ \bar{t}_1 \cdot \overline{\theta} &= (r+u)(1+\phi_{,\theta}) \\ \bar{t}_1 \cdot \overline{k} &= z_{,\theta} + w_{,\theta} \end{aligned}$$

Thus the unit tangent vector to the face $r = constant$ is

$$\overline{T}_1 = \frac{u_{,\theta}\,\bar{r} + (r+u)(1+\phi_{,\theta})\,\overline{\theta} + (z_{,\theta}+w_{,\theta})\,\overline{k}}{\sqrt{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}} \qquad (2.16)$$

Similarly, we can define a tangent to the face $\theta = constant$ using equations (2.7)–(2.9).

$$\bar{t}_2 = \frac{dx^*}{dr}\,\bar{\imath} + \frac{dy^*}{dr}\,\bar{\jmath} + \frac{dz^*}{dr}\,\overline{k}$$

The components of this vector in the $\bar{r}$, $\overline{\theta}$, and $\overline{k}$ directions are

$$\begin{aligned} \bar{t}_2 \cdot \bar{r} &= 1 + u_{,r} \\ \bar{t}_2 \cdot \overline{\theta} &= (r+u)\phi_{,r} \\ \bar{t}_2 \cdot \overline{k} &= z_{,r} + w_{,r} \end{aligned}$$

Thus the unit tangent vector to the face $\theta = constant$ is

$$\overline{T}_2 = \frac{(1+u_{,r})\,\bar{r} + (r+u)\phi_{,r}\,\overline{\theta} + (z_{,r}+w_{,r})\,\overline{k}}{\sqrt{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}} \qquad (2.17)$$

Unit Normal Vectors We derive the unit normal vectors by first noticing that in the absence of shear strain, $\overline{T}_1$ is tangent to the face $r = constant$ and normal to the face $\theta = constant$. Likewise, $\overline{T}_2$ is normal to the face $r = constant$. It is in the presence of shear strain that these vectors no longer can double as unit normals. Thus we shall derive the unit normals as rotations of the unit tangents through the angle γ.

From equation (2.15) we can determine the following relations

$$\sin\gamma = \frac{\frac{u_{,\theta}(1+u_{,r})}{r+u} + (r+u)(1+\phi_{,\theta})\phi_{,r}}{\frac{u_{,\theta}^2(1+u_{,r})^2}{(r+u)^2} + (r+u)^2(1+\phi_{,\theta})^2\phi_{,r}^2 + (1+\phi_{,\theta})^2(1+u_{,r})^2 + u_{,\theta}^2\phi_{,r}^2} \tag{2.18}$$

$$\cos\gamma = \frac{(1+\phi_{,\theta})(1+u_{,r}) - u_{,\theta}\phi_{,r}}{\frac{u_{,\theta}^2(1+u_{,r})^2}{(r+u)^2} + (r+u)^2(1+\phi_{,\theta})^2\phi_{,r}^2 + (1+\phi_{,\theta})^2(1+u_{,r})^2 + u_{,\theta}^2\phi_{,r}^2} \tag{2.19}$$

We obtain the unit normal vector to the face $r = constant$ by rotating $\overline{T}_2$ through the angle $-\gamma$ in the $\overline{r}\overline{\theta}$-plane.

$$\overline{N}_1 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{T}_2 \tag{2.20}$$

Similarly, we may obtain the unit normal vector to the face $\theta = constant$ by rotating $\overline{T}_1$ through the angle γ in the $\overline{r}\overline{\theta}$-plane.

$$\overline{N}_2 = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{T}_1 \tag{2.21}$$

It can be shown that the projections of $\overline{N}_1$ and $\overline{T}_1$ into the $\overline{r}\overline{\theta}$-plane are orthogonal, as we would expect. The same is true of $\overline{N}_2$ and $\overline{T}_2$.

Forces to Be Considered Consider an element of the deformed membrane as shown in Figure 2.1. The vectors indicated are drawn on the positive faces. Table 2.1 outlines the elements which determine the magnitude of the forces. Note that the vertical pressure acts upon the undeformed surface area of the element, whereas the stresses σ_r and σ_θ act upon the deformed surface areas of their respective faces. Also, we introduce here the shear stress, τ, which is a function of the shear strain, γ, such that $\tau = 0$ when $\gamma = 0$.

Equilibrium Conditions The deformed state occurs when all forces are in equilibrium. This implies that the sum of all forces described in section 2.2.1 is the zero

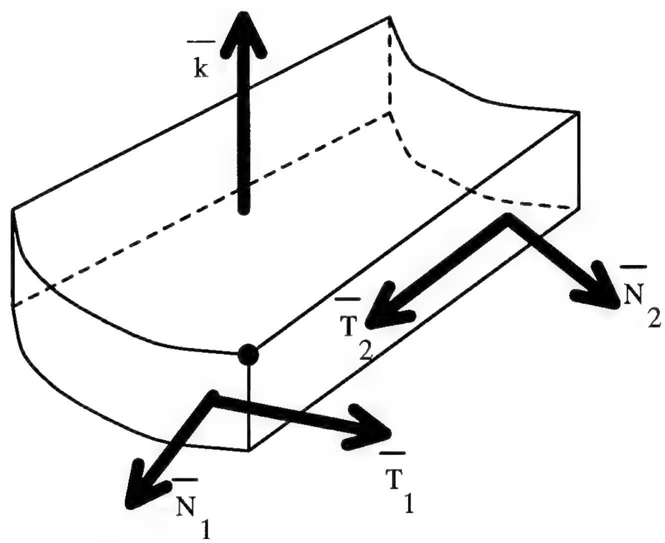

Figure 2.1 An element of the deformed membrane with associated unit vectors. Vectors are drawn on the positive faces with which they are associated. However, all vectors act through the darkened point $(r+u, \theta+\phi)$.

vector. The forces are evaluated on the four lateral faces as well as the top face. Hence, the equilibrium condition is

$$\begin{aligned}
&\left[(\sigma_r \overline{N}_1 + \tau \overline{T}_1) h \,\Delta\theta \sqrt{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}\right]\Big|_r^{r+\Delta r} \\
&+ \left[(\sigma_\theta \overline{N}_2 + \tau \overline{T}_2) h \,\Delta r \sqrt{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}\right]\Big|_\theta^{\theta+\Delta\theta} \\
&\qquad + P\left(r\,\Delta r\,\Delta\theta + \frac{(\Delta r)^2}{2}\Delta\theta\right)\overline{k} \;=\; \overline{0}
\end{aligned} \tag{2.22}$$

Dividing through by $h\,\Delta r\,\Delta\theta$ and taking the limit as Δr and $\Delta\theta$ go to zero yields

$$\begin{aligned}
&\frac{\partial}{\partial r}\left\{(\sigma_r \overline{N}_1 + \tau \overline{T}_1)\sqrt{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}\right\} \\
&+\frac{\partial}{\partial \theta}\left\{(\sigma_\theta \overline{N}_2 + \tau \overline{T}_2)\sqrt{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}\right\} + \frac{Pr}{h}\overline{k} \;=\; \overline{0}
\end{aligned} \tag{2.23}$$

For convenience, we now define the following scalars:

$$d_1 = \sqrt{\frac{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}} \tag{2.24}$$

$$d_2 = \sqrt{\frac{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + (z_{,\theta}+w_{,\theta})^2}} \tag{2.25}$$

Stress	Area of Face	Direction
P	$r\,\Delta r\,\Delta\theta + \frac{(\Delta r)^2}{2}\,\Delta\theta$	$\overline{k}$
σ_r	$h\,\Delta\theta\sqrt{u_{,\theta}^2 + (r+u)^2(1+\phi_{,\theta})^2 + w_{,\theta}^2}$	$\overline{N}_1$
τ		$\overline{T}_1$
σ_θ	$h\,\Delta r\sqrt{(1+u_{,r})^2 + (r+u)^2\phi_{,r}^2 + (z_{,r}+w_{,r})^2}$	$\overline{N}_2$
τ		$\overline{T}_2$

Table 2.1 Factors determining the force vectors acting on the membrane

Substituting in the equations for the unit normal and tangent vectors, we then may rewrite equation (2.23) as

$$\begin{aligned}
&\frac{\partial}{\partial r}\{\sigma_r d_1[(1+u_{,r})\cos\gamma + (r+u)\phi_{,r}\sin\gamma]\,\overline{r}\} \\
&\quad - \frac{\partial}{\partial r}\left\{\sigma_r d_1[(1+u_{,r})\sin\gamma - (r+u)\phi_{,r}\cos\gamma]\,\overline{\theta}\right\} + \frac{\partial}{\partial r}\left\{\sigma_r d_1(z_{,r}+w_{,r})\,\overline{k}\right\} \\
&\quad + \frac{\partial}{\partial r}\left\{\tau[u_{,\theta}\,\overline{r} + (r+u)(1+\phi_{,\theta})\,\overline{\theta} + (z_{,\theta}+w_{,\theta})\,\overline{k}]\right\} \\
&\quad + \frac{\partial}{\partial \theta}\{\sigma_\theta d_2[u_{,\theta}\cos\gamma - (r+u)(1+\phi_{,\theta})\sin\gamma]\,\overline{r}\} \\
&\quad + \frac{\partial}{\partial \theta}\left\{\sigma_\theta d_2[u_{,\theta}\sin\gamma + (r+u)(1+\phi_{,\theta})\cos\gamma]\,\overline{\theta}\right\} + \frac{\partial}{\partial \theta}\left\{\sigma_\theta d_2(z_{,\theta}+w_{,\theta})\,\overline{k}\right\} \\
&\quad + \frac{\partial}{\partial \theta}\left\{\tau[(1+u_{,r})\,\overline{r} + (r+u)\phi_{,r}\,\overline{\theta} + (z_{,r}+w_{,r})\,\overline{k}]\right\} + \frac{Pr}{h}\,\overline{k} = \overline{0}
\end{aligned} \tag{2.26}$$

Recall that $\overline{r}$ and $\overline{\theta}$ as defined in equations (2.11) and (2.12) are functions of r and θ. Thus we cannot simply pull them out of the differentiation in order to sum forces in each of the component directions. We must use the product rule before we can sum forces.

Note that $\overline{r}$ and $\overline{\theta}$ and their derivatives satisfy the following relations

$$\begin{aligned}
\frac{\partial\overline{r}}{\partial r} &= \phi_{,r}\,\overline{\theta} \\
\frac{\partial\overline{r}}{\partial \theta} &= (1+\phi_{,\theta})\,\overline{\theta} \\
\frac{\partial\overline{\theta}}{\partial r} &= -\phi_{,r}\,\overline{r} \\
\frac{\partial\overline{\theta}}{\partial \theta} &= -(1+\phi_{,\theta})\,\overline{r}
\end{aligned}$$

Using these relations, performing the differentiation on the vectors yields the equation

$$
\begin{aligned}
&\sigma_r d_1[(1+u_{,r})\cos\gamma + (r+u)\phi_{,r}\sin\gamma]\phi_{,r}\,\overline{\theta} \\
&\quad + \frac{\partial}{\partial r}\left\{\sigma_r d_1[(1+u_{,r})\cos\gamma + (r+u)\phi_{,r}\sin\gamma]\right\}\overline{r} \\
&\quad + \sigma_r d_1[(1+u_{,r})\sin\gamma - (r+u)\phi_{,r}\cos\gamma]\phi_{,r}\,\overline{r} \\
&\quad - \frac{\partial}{\partial r}\left\{\sigma_r d_1[(1+u_{,r})\sin\gamma - (r+u)\phi_{,r}\cos\gamma]\right\}\overline{\theta} \\
&\quad + \frac{\partial}{\partial r}\left\{\sigma_r d_1(z_{,r}+w_{,r})\right\}\overline{k} + \tau(r+u)u_{,\theta}\phi_{,r}\,\overline{\theta} + \frac{\partial}{\partial r}\left\{\tau u_{,\theta}\right\}\overline{r} \\
&\quad - \tau(r+u)(1+\phi_{,\theta})\phi_{,r}\,\overline{r} + \frac{\partial}{\partial r}\left\{\tau(r+u)(1+\phi_{,\theta})\right\}\overline{\theta} + \frac{\partial}{\partial r}\left\{\tau(z_{,\theta}+w_{,\theta})\right\}\overline{k} \\
&\quad + \sigma_\theta d_2[u_{,\theta}\cos\gamma - (r+u)(1+\phi_{,\theta})\sin\gamma](1+\phi_{,\theta})\,\overline{\theta} \\
&\quad + \frac{\partial}{\partial\theta}\left\{\sigma_\theta d_2[u_{,\theta}\cos\gamma - (r+u)(1+\phi_{,\theta})\sin\gamma]\right\}\overline{r} \\
&\quad - \sigma_\theta d_2[u_{,\theta}\sin\gamma + (r+u)(1+\phi_{,\theta})\cos\gamma](1+\phi_{,\theta})\,\overline{r} \\
&\quad + \frac{\partial}{\partial\theta}\left\{\sigma_\theta d_2[u_{,\theta}\sin\gamma + (r+u)(1+\phi_{,\theta})\cos\gamma]\right\}\overline{\theta} \\
&\quad + \frac{\partial}{\partial\theta}\left\{\sigma_\theta d_2(z_{,\theta}+w_{,\theta})\right\}\overline{k} + \tau(1+u_{,r})(1+\phi_{,\theta})\,\overline{\theta} + \frac{\partial}{\partial\theta}\left\{\tau(1+u_{,r})\right\}\overline{r} \\
&\quad - \tau(r+u)(1+\phi_{,\theta})\phi_{,r}\,\overline{r} + \frac{\partial}{\partial\theta}\left\{\tau(r+u)\phi_{,r}\right\}\overline{\theta} + \frac{\partial}{\partial\theta}\left\{\tau(w_{,r}+z_{,r})\right\}\overline{k} + \frac{Pr}{h}\overline{k} = \overline{0}
\end{aligned}
\tag{2.27}
$$

Separating into the components of this equation yields our system of three partial differential equations for the radial, vertical, and angular displacements of our non-axisymmetric deformation.

$$
\begin{aligned}
&\frac{\partial}{\partial r}\left\{\sigma_r d_1[(1+u_{,r})\cos\gamma + (r+u)\phi_{,r}\sin\gamma] + \tau u_{,\theta}\right\} \\
&\quad + \frac{\partial}{\partial\theta}\left\{\sigma_\theta d_2[u_{,\theta}\cos\gamma - (r+u)(1+\phi_{,\theta})\sin\gamma] + \tau(1+u_{,r})\right\} \\
&\quad + \sigma_r d_1[(1+u_{,r})\sin\gamma - (r+u)\phi_{,r}\cos\gamma]\phi_{,r} - \tau(r+u)(1+\phi_{,\theta})\phi_{,r} \\
&\quad - \sigma_\theta d_2[u_{,\theta}\sin\gamma + (r+u)(1+\phi_{,\theta})\cos\gamma](1+\phi_{,\theta}) - \tau(r+u)(1+\phi_{,\theta})\phi_{,r} = 0
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial r}\left\{\sigma_r d_1[(1+u_{,r})\sin\gamma - (r+u)\phi_{,r}\cos\gamma] - \tau(r+u)(1+\phi_{,\theta})\right\} \\
&\quad + \frac{\partial}{\partial\theta}\left\{\sigma_\theta d_2[u_{,\theta}\sin\gamma + (r+u)(1+\phi_{,\theta})\cos\gamma] + \tau(r+u)\phi_{,r}\right\} \\
&\quad + \sigma_r d_1[(1+u_{,r})\cos\gamma + (r+u)\phi_{,r}\sin\gamma]\phi_{,r} + \tau(r+u)u_{,\theta}\phi_{,r} \\
&\quad + \sigma_\theta d_2[u_{,\theta}\cos\gamma - (r+u)(1+\phi_{,\theta})\sin\gamma](1+\phi_{,\theta}) + \tau(1+u_{,r})(1+\phi_{,\theta}) = 0
\end{aligned}
\tag{2.29}
$$

$$\frac{\partial}{\partial r}\left\{\sigma_r d_1(z_{,r}+w_{,r})+\tau(z_{,\theta}+w_{,\theta})\right\} + \frac{\partial}{\partial \theta}\left\{\sigma_\theta d_2(z_{,\theta}+w_{,\theta})+\tau(1+u_{,r})\right\}+\frac{Pr}{h}=0 \tag{2.30}$$

with boundary conditions as specified by (2.1). Notice that the result is a nonlinear system of boundary value problems. As a result, the questions of existence and uniqueness of solutions are not easily answered. Antman [1, pgs 460, 469] provides some discussion and references concerning existence and uniqueness in the context of three dimensional elasticity.

2.2.2 Form of the Constitutive Law

Throughout our derivation, we have not discussed the form of the constitutive law. In considering the appropriate relationship, it is appropriate to recall the structure of the problem, as illustrated by Figure 1.1. Our model certainly allows for the heterogeneous composition of the material. We consider two approaches for accounting for this structure in the form of the constitutive law.

Treating the Membrane as Macroscopically Homogeneous

Although we may not wish to consider the heterogeneity of the membrane directly, we can use some knowledge of the underlying structure to decide which constitutive relationship we should consider. The entire diaphragm should not be expected to respond the same to radial strain as it would to circumferential strain, because the periodic structure occurs in the circumferential direction only [2]. This indicates that we should consider the membrane to have two orthogonal directions of elastic symmetry—the radial direction and the circumferential direction. Such a material is called orthotropic. Furthermore, since the directions of symmetry of this material coincide with the basis vectors which we use in our model, we consider the membrane to be specially orthotropic. This allows us to express the stress-strain relationship in a much simpler form. If we use a linear constitutive law, the stress-strain relationship is

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \end{pmatrix} = \frac{1}{1-\nu_{r\theta}\nu_{\theta r}} \begin{pmatrix} E_r & \nu_{\theta r}E_r \\ \nu_{r\theta}E_\theta & E_\theta \end{pmatrix} \begin{pmatrix} \mathcal{E}_r \\ \mathcal{E}_\theta \end{pmatrix}. \tag{2.31}$$

Here E_r and $\nu_{r\theta}$ are the Young's Modulus and Poisson Ratio associated with strain applied in the radial direction. Similarly, E_θ and $\nu_{\theta r}$ are the moduli associated with

circumferential stress. The matrix in this stress-strain relationship is called the stiffness matrix. At first glance, there appears to be four moduli that must be recovered. However, under the additional assumption of the existence of a strain energy density function, we may find a relationship between the moduli which allows us to compute one in terms of the others. A strain energy density function, W is a function such that the stresses are derived according to the equations

$$\sigma_r = \frac{\partial W}{\partial \mathcal{E}_r}, \ \sigma_\theta = \frac{\partial W}{\partial \mathcal{E}_\theta}.$$

A material for which there exists such a function is called hyperelastic. It can be shown [8] that under this assumption, the matrix in equation (2.31) must be symmetric. Hence we find that

$$\begin{aligned} \nu_{\theta r} E_r &= \nu_{r\theta} E_\theta \\ \Longleftrightarrow \nu_{\theta r} &= \frac{E_\theta}{E_r} \nu_{r\theta}. \end{aligned} \tag{2.32}$$

Thus we have only three elastic moduli to recover if we consider the membrane to be homogeneous but orthotropic. Hyperelasticity and orthotropy are common assumptions in the literature regarding stress-strain laws for muscle and other biological tissues [4, 7, 13, 14, 21, 22].

It should be noted that a special case of an orthotropic material is when the material responds to a strain in exactly the same way, regardless of the orientation of the material. The material would then be called isotropic. As we have mentioned, the assumption of isotropy is not realistic for our problem because the materials which compose our membrane are not likely to respond identically to strains. For an isotropic material, the stiffness matrix is simplified to yield the following stress-strain relationship:

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \end{pmatrix} = \frac{1}{1-\nu^2} \begin{pmatrix} E & \nu E \\ \nu E & E \end{pmatrix} \begin{pmatrix} \mathcal{E}_r \\ \mathcal{E}_\theta \end{pmatrix}. \tag{2.33}$$

Here E is the Young's Modulus and ν is the Poisson Ratio. Thus we have only two elastic moduli to recover for an isotropic material.

Treating the Membrane as a Unidirectional Fiber Composite

Another approach to considering the heterogeneity of the membrane is to look to the mechanics of composite materials. Our observation of the structure of the diaphragm

in chapter 1 directs our interest to the study of unidirectional fiber composites. Under this approach, the muscle bundles would be considered the fibers which are embedded in a collagen matrix. The biggest difference between this approach and the use of an orthotropic constitutive law is that here we can directly consider not only the structure of the muscle, but also the properties of the individual materials which make up the membrane, i.e. the muscle bundles and the collagen matrix. Once we have arrived at some representative volume element of the composite sheet, we can use volume averaged stresses which are related to the volume averaged strains via some effective moduli of an equivalent homogeneous material. The process of arriving at these effective moduli can be quite difficult, however.

Gibson [8] outlines two general methods to approach the problem. The first is a "theory of elasticity" approach in which the equations of elasticity must be satisfied everywhere in the model. No simplifying assumptions are made concerning stress and strain distributions. As a result, solutions of any sort are difficult to obtain. However, several analytical and numerical solutions of this type can be found in the literature [11, 3, 5, 10].

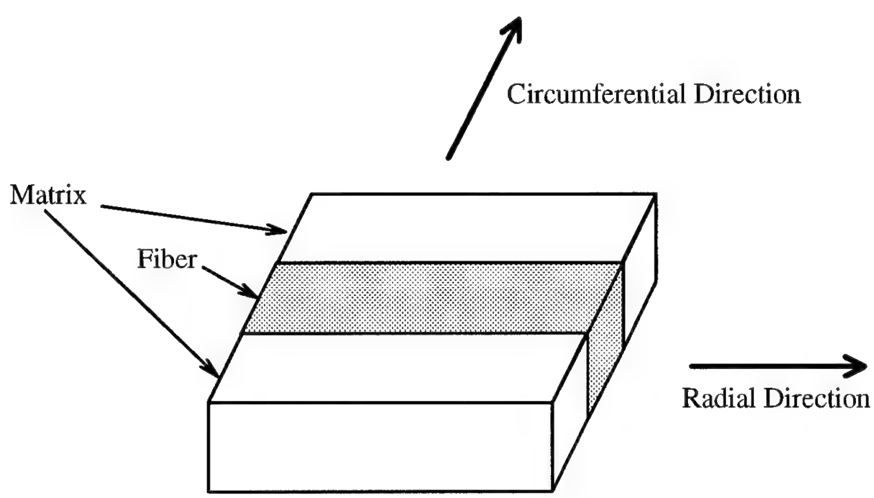

Figure 2.2 Representative volume element of a unidirectional fiber composite used by Gibson, as well as Tsai and Hahn in deriving expressions for the effective moduli of the composite.

The second approach is called the "mechanics of materials" approach. Several simplifying assumptions are made in this approach so that the details of the stress

and strain distributions are unnecessary. In addition, the fiber-packing geometry is often unspecified. These assumptions are:

1. Fiber orientation does not change from that indicated in the representative volume element
2. Dimensions of the fiber and matrix do not change along the length of the element
3. The matrix and fiber are perfectly bonded
4. Poisson and shear strains between the matrix and fiber are neglected
5. For longitudinal (i.e. radial) stress, average strains in the composite, fiber, and matrix are equal
6. For transverse (i.e. circumferential) stress, stresses in the composite, fiber, and matrix are equal

The first two assumptions allow us to express the fiber-packing geometry exclusively in terms of the volume fraction of the fiber, v_f and the volume fraction of the matrix, v_m, where we assume that $v_f + v_m = 1$. The remaining assumptions greatly simplify the state of stress of the representative volume element. Using these assumptions, Gibson derives expressions for the effective moduli of the composite for a material whose representative volume element is shown in Figure 2.2, where the fiber is orthotropic with isotropic matrix. Tsai and Hahn [20] derive the equivalent expressions for the case where both the matrix and the fiber are isotropic. Since the equations presented by Tsai and Hahn are for a special case of those derived by Gibson, we present those equations.

$$E_r = v_f(E_f)_r + v_m E_m \tag{2.34}$$

$$\frac{1}{E_\theta} = \frac{v_f}{(E_f)_\theta} + \frac{v_m}{E_m} \tag{2.35}$$

$$\nu_{r\theta} = v_f(\nu_f)_{r\theta} + v_m \nu_m \tag{2.36}$$

Here the subscripts f and m refer to the fiber and matrix, respectively. Recall that due to the assumption of hyperelasticity, it is not necessary to specify a separate equation for $\nu_{\theta r}$ of the composite. Note that E_r and $\nu_{r\theta}$ are computed by a rule of mixtures, while E_θ is computed by an inverse rule of mixtures. Empirical data seem to verify that equation (2.34) is a good model of the properties of the composite material. However, equation (2.35) often fails to match empirical data sufficiently [8,

20]. This is due to assumption 6. Due to equilibrium constraints, this assumption is necessarily true when the matrix and fiber blocks have equal areas normal to the circumferential direction. Unfortunately, few composite materials have this property. There are methods of correcting this error, but most require additional knowledge of the fiber packing structure. Hopkins and Chamis [12] present a refined model for transverse properties using a square fiber-packing array and a method of dividing the representative volume element into subregions. Tsai and Hahn [20] attempt to remedy the model by allowing a Poisson strain caused by transverse loading. For our problem, we will assume the simplest case to be true. Therefore, we will use equation (2.35) in computing the transverse Young's modulus for the composite.

2.3 Homogenization of the Model

Equations (2.28)–(2.30) give us a very general model of membrane deformation. It is within the scope of this model to allow the material properties of the membrane to be different at every point (r, θ) in the domain of definition. We, however, are concerned with a much more regular and periodic structure. As we noted earlier, we assume the muscle and connective tissue in the diaphragm to occur in a periodic fashion. We would like to use this knowledge of a much simpler structure to simplify the system of partial differential equations we developed in the previous section. Homogenization gives us a tool to find some average behavior over the range of θ values. Having determined this behavior, we can then consider the membrane to be homogeneous in the circumferential direction. The implications of this homogeneity would greatly simplify our equilibrium equations.

Parton and Kudryavtsev [19] present formal mathematical homogenization techniques both for periodic structures, as we are considering here, and for regions with a wavy boundary. Oleinik, Shamev, and Yosifian [18] discuss similar methods for several problems in elasticity. While these methods have a solid theoretical basis, application of these methods to a system of partial differential equations could be difficult and tedious. We therefore consider the use of effective moduli to homogenize the constitutive law in θ. If we assume that the structure of the muscle and collagen is similar to that shown in figure 2.2, then we can use the "mechanics of materials" approach discussed in section 2.2.2 to compute the effective moduli. With these effective moduli, we can then consider the membrane to be macroscopically homogeneous in the circumferential direction. We homogenize the boundary by simply redefining

our domain. If we let $f_1(\theta) \equiv a$ and $f_2(\theta) \equiv b$ for $a < b$, our domain becomes

$$\Omega = \{(r,\theta) : r \in (a,b), \theta \in [0, 2\pi)\}$$

If we also redefine the function z to be a function of r alone, these homogenizations will have the following effects on our model:

1. There will be no circumferential displacement. Thus $\phi \equiv 0$.

2. The radial and vertical displacements, as well as the radial and circumferential stresses, will no longer vary with θ. Thus $u = u(r)$, $w = w(r)$, $\sigma_r = \sigma_r(r)$, and $\sigma_\theta = \sigma_\theta(r)$.

3. By equation (2.15), these first two facts imply that the shear strain, γ, is also identically zero. Thus $\tau \equiv 0$.

4. Since there is no more θ dependence, all partial derivatives with respect to θ are zero. The partial derivatives with respect to r become full derivatives.

As a consequence of fact 4, we will replace the subscript notation of differentiation with $' = \frac{d}{dr}$. Using these facts, the tensile strains defined in equations (2.6) and (2.10) become

$$\begin{aligned} \mathcal{E}_\theta &= \frac{u}{r}\,, && (2.37) \\ \mathcal{E}_r &= \sqrt{\frac{(1+u')^2 + (z'+w')^2}{1+(z')^2}} - 1\,. && (2.38) \end{aligned}$$

The scalars defined by equations (2.24)–(2.25) become

$$d_1 = \frac{r+u}{\sqrt{(1+u')^2 + (z'+w')^2}} = \frac{1}{d_2}.$$

Equation (2.29) becomes a tautology, while the remaining equilibrium equations simplify to

$$\begin{aligned} \frac{d}{dr}\left\{\frac{\sigma_r(r+u)(1+u')}{\sqrt{(1+u')^2+(z'+w')^2}}\right\} - \sigma_\theta\sqrt{(1+u')^2+(z'+w')^2} &= 0 && (2.39) \\ \frac{d}{dr}\left\{\frac{\sigma_r(r+u)(z'+w')}{\sqrt{(1+u')^2+(z'+w')^2}}\right\} + \frac{Pr}{h} &= 0 && (2.40) \end{aligned}$$

with either the displacement boundary conditions

$$\begin{aligned} u(a) &= \alpha_u, \quad u(b) = \beta_u, \\ w(a) &= \alpha_w, \quad w(b) = \beta_w, \end{aligned} \tag{2.41}$$

or the traction boundary conditions

$$\begin{aligned} \sigma_r(a) &= \alpha_\sigma, \quad \sigma_r(b) = \beta_\sigma, \\ w(a) &= \alpha_w, \quad w(b) = \beta_w, \end{aligned} \tag{2.42}$$

where $\alpha_u, \alpha_w, \alpha_\sigma, \beta_u, \beta_w$, and β_σ are constants. Recall that σ_r and σ_θ obey the orthotropic constitutive law of equation (2.31) with the elastic moduli then defined by equations (2.34)–(2.36). Note that if we allow σ_r and σ_θ to be general functions of the strains, equations (2.39)–(2.40) are very similar to those presented by Dickey [6]. However, we have now extended them to allow non-flat reference states where the reference vertical displacement can be written as a function of r.

Chapter 3

Resolution of the Forward Problem

When we know the properties of the material, the vertical pressure P, and the membrane thickness, h, the only remaining unknowns in (2.39)–(2.40) are radial displacement, u, and the vertical displacement, w. We would now like to consider how to solve these equations for u and w. This problem is called the forward problem. While our goal is to solve for the material properties when the displacements are known, i.e. the inverse problem, knowledge of how to solve the forward problem is essential to solution of the inverse problem.

In section 3.1 we look at how to solve equations (2.39)–(2.40) in terms of the second derivatives of the displacements, which is an essential step for some numerical methods which we consider in section 3.2. We then look at some examples of numerical solutions of equations (2.39)–(2.40) with different boundary conditions and various choices of Young's moduli and Poisson ratio.

3.1 Solving the System of Equations for u'' and w''

Our first step is to rewrite equations (2.39)–(2.40) so as to solve for u'' and w''. We then would have a second order system of differential equations of the form

$$\begin{aligned} u'' &= f(r, u, w, u', w') \\ w'' &= g(r, u, w, u', w') \end{aligned} \tag{3.1}$$

If we can derive a system of this form, then the solution of the forward problem will simply be the solution of a two point boundary value problem, as is generally the case for a second order differential equation. Finding the equations f and g is not a straightforward task. The second derivatives of u and w arise in (2.39)–(2.40) from carrying out the differentiation in terms of r. It is clear that differentiating terms such as $(1 + u')$ result in second derivatives. However, they also arise in the differentiation of σ_r, since σ_r also depends on u' and w'. We therefore must be able to separate this dependence in order to solve the equations for u'' and w''.

We see that σ_r depends on u' and w' since it is defined to be a function of $\mathcal{E}_r$. Recall that in the case of linear constitutive laws, such as those presented in section 2.2.2, we can write

$$\sigma_r = q_1^T \begin{pmatrix} \mathcal{E}_r \\ \mathcal{E}_\theta \end{pmatrix},$$

where q_1^T is the first row of the stiffness matrix. Hence

$$\sigma_r' = {q_1'}^T \begin{pmatrix} \mathcal{E}_r \\ \mathcal{E}_\theta \end{pmatrix} + q_1^T \begin{pmatrix} \mathcal{E}_r' \\ \mathcal{E}_\theta' \end{pmatrix},$$

with

$$\begin{aligned} \mathcal{E}_r' &= \frac{(1+u')u'' + (z'+w')(z''+w'')}{\sqrt{[1+(z')^2][(1+u')^2+(z'+w')^2]}} \\ &\quad - \frac{z'z''\sqrt{(1+u')^2+(z'+w')^2}}{[1+(z')^2]^{3/2}}. \end{aligned} \tag{3.2}$$

Since the stiffness matrix has no dependence on u' or w', the first term of the above sum has no dependence on u'' and w''. The same applies to any term involving $\mathcal{E}_\theta'$. From (3.2), we see that $\mathcal{E}_r'$ is affine in u'' and w''. Thus we can write

$$\sigma_r' = s_u u'' + s_w w'' + s,$$

where

$$\begin{aligned} s_u &= q_1^T \begin{pmatrix} \frac{1+u'}{\sqrt{[1+(z')^2][(1+u')^2+(z'+w')^2]}} \\ 0 \end{pmatrix} \\ s_w &= q_1^T \begin{pmatrix} \frac{z'+w'}{\sqrt{[1+(z')^2][(1+u')^2+(z'+w')^2]}} \\ 0 \end{pmatrix} \\ s &= {q_1'}^T \begin{pmatrix} \mathcal{E}_r \\ \mathcal{E}_\theta \end{pmatrix} + q_1^T \begin{pmatrix} \frac{(z'+w')z''}{\sqrt{[1+(z')^2][(1+u')^2+(z'+w')^2]}} - \frac{z'z''\sqrt{(1+u')^2+(z'+w')^2}}{[1+(z')^2]^{3/2}} \\ \frac{u'}{r} - \frac{u}{r^2} \end{pmatrix} \end{aligned}$$

Using this fact, we can then rewrite equations (2.39)–(2.40) in the form

$$A \begin{pmatrix} u'' \\ w'' \end{pmatrix} = b,$$

where

$$A = \begin{pmatrix} \frac{\sigma_r(r+u)+(r+u)(1+u')s_u}{d} - \frac{\sigma_r(r+u)(1+u')^2}{d^3} & -\frac{\sigma_r(r+u)(1+u')(z'+w')}{d^3} + \frac{(r+u)(1+u')s_w}{d} \\ -\frac{\sigma_r(r+u)(1+u')(z'+w')}{d^3} + \frac{(r+u)(z'+w')s_u}{d} & \frac{\sigma_r(r+u)+(r+u)(z'+w')s_w}{d} - \frac{\sigma_r(r+u)(z'+w')^2}{d^3} \end{pmatrix} \tag{3.3}$$

$$b = \begin{pmatrix} \sigma_\theta d - \frac{\sigma_r(1+u')^2-(r+u)(1+u')s}{d} + \frac{\sigma_r(r+u)(1+u')(z'+w')z''}{d^3} \\ -\frac{P_T}{h} - \frac{\sigma_r(1+u')(z'+w')-(r+u)(z'+w')s-\sigma_r(r+u)z''}{d} + \frac{\sigma_r(r+u)(z'+w')^2z''}{d^3} \end{pmatrix} \tag{3.4}$$

with

$$d = \sqrt{(1+u')^2 + (z'+w')^2}\,.$$

If this matrix A is invertible, then we can write equations (2.39)–(2.40) in the form of (3.1), and our second order system of differential equations would be

$$\begin{pmatrix} u'' \\ w'' \end{pmatrix} = A^{-1}b\,. \tag{3.5}$$

with boundary conditions (2.41) or (2.42). Of course, A is invertible if and only if $\det A \neq 0$. We can see that if $r + u = 0$, the determinant would equal zero. This would only happen if a point in the membrane was deformed radially to the origin. Under a vertical pressure, such a deformation is not likely to occur. Another more interesting circumstance which would make A singular is if $\sigma_r = 0$. In the case of the linear constitutive laws, this is only possible if $\mathcal{E}_r$ and $\mathcal{E}_\theta$ both equal zero, since the stiffness matrix is necessarily nonzero. A simple example of where this would occur is when $u = u' = w' = z' = 0$. While this is not likely to occur in the actual deformed state, it is important in the numerical solution of this problem to avoid this singularity—especially as an initial guess. Another place where discontinuity occurs is where $\sqrt{(1+u')^2 + (z'+w')^2}$ vanishes. Other than these singularities, $A^{-1}b$ is continuous. Keller [15] gives conditions which ensure that solutions to this system exist and are unique.

3.2 Numerical Solution

Note that the system of equations (3.5) is highly nonlinear in u and w. Thus we must use numerical methods for the solution of nonlinear systems of ordinary differential equations. Two methods which can be used for problems of this type are the shooting method and the finite difference method.

3.2.1 Nonlinear Shooting Method

Using a standard change of variables, (3.5) can be transformed into a first order system of ordinary differential equations. After this change of variables, a Runge-Kutta method can be used to integrate the system from a set of initial values of u, w, u', and w'. However, we do not know that the resulting solution will meet boundary conditions at endpoint. Recall the boundary conditions of equation (2.41). The shooting method starts the Runge-Kutta method with initial conditions $u(a) = \alpha_u$, $w(a) = \alpha_w$ and iteratively selects values of $u'(a)$ and $w'(a)$ so that the integration returns values of $u(b)$ and $w(b)$ that are within a certain tolerance of β_u and β_w, respectively. The code we have implemented uses a secant update of these initial derivative values. As a result, quite a few iterations are required for convergence, but the method does consistently converge. One useful feature of the shooting method is that its accuracy is the same as the accuracy of the integration method. Thus, if we use a fourth-order Runge-Kutta method, our shooting method is fourth-order accurate. It is therefore relatively simple to improve the accuracy of the overall shooting method.

For more general boundary conditions, we follow a similar scheme. We search for initial values of u, w, u', and w' which yield a function which matches the four boundary conditions. Hence we have a system of four nonlinear equations in $u(a)$, $w(a)$, $u'(a)$, and $w'(a)$. This system can then be solved numerically for the proper initial values. This approach allows for the boundary conditions to be nonlinear in u, w, u', and w'. Therefore, we must use this scheme in the case of the traction boundary conditions of equation (2.42). Keller [15, pg 50] gives conditions which ensure that Newton's method will converge uniquely in the numerical solution of this system.

3.2.2 Nonlinear Finite Difference Method

The finite difference method uses finite difference approximations to the derivatives which appear in (3.5), resulting in a system of nonlinear equations in the values of u and w at the mesh points of the partition

$$a = r_0 < r_1 < \cdots < r_n < r_{n+1} = b.$$

We define $u_i = u(r_i)$ and $w_i = w(r_i)$ to be these values, with boundary conditions $u_0 = \alpha_u$, $w_0 = \alpha_w$, $u_{n+1} = \beta_u$, $w_{n+1} = \beta_w$ as specified by (2.41). For simplicity, let us

also assume a uniform mesh, i.e. $r_{i+1} - r_i = \Delta r$, for $i = 1, 2, \ldots, n$. Recall that with equation (3.5) we have defined the functions f and g of equation (3.1) such that

$$\begin{pmatrix} f(r, u, w, u', w') \\ g(r, u, w, u', w') \end{pmatrix} = A^{-1}b.$$

Applying the second order finite difference approximations to the derivatives, we have the following system of $2n$ equations:

$$\begin{aligned}
-\alpha_u + 2u_1 - u_2 + (\Delta r)^2 f(r_1, u_1, w_1, \frac{u_2 - \alpha_u}{h}, \frac{w_2 - \alpha_w}{h}) &= 0, \\
-u_1 + 2u_2 - u_3 + (\Delta r)^2 f(r_2, u_2, w_2, \frac{u_3 - u_1}{h}, \frac{w_3 - w_1}{h}) &= 0, \\
&\vdots \\
-u_{n-2} + 2u_{n-1} - u_n + (\Delta r)^2 f(r_{n-1}, u_{n-1}, w_{n-1}, \frac{u_n - u_{n-2}}{h}, \frac{w_n - w_{n-2}}{h}) &= 0, \\
-u_{n-1} + 2u_n - \beta_u + (\Delta r)^2 f(r_n, u_n, w_n, \frac{\beta_u - u_{n-1}}{h}, \frac{\beta_w - w_{n-1}}{h}) &= 0, \\
-\alpha_w + 2w_1 - w_2 + (\Delta r)^2 g(r_1, u_1, w_1, \frac{u_2 - \alpha_u}{h}, \frac{w_2 - \alpha_w}{h}) &= 0, \\
-w_1 + 2w_2 - w_3 + (\Delta r)^2 g(r_2, u_2, w_2, \frac{u_3 - u_1}{h}, \frac{w_3 - w_1}{h}) &= 0, \\
&\vdots \\
-w_{n-2} + 2w_{n-1} - w_n + (\Delta r)^2 g(r_{n-1}, u_{n-1}, w_{n-1}, \frac{u_n - u_{n-2}}{h}, \frac{w_n - w_{n-2}}{h}) &= 0, \\
-w_{n-1} + 2w_n - \beta_w + (\Delta r)^2 g(r_n, u_n, w_n, \frac{\beta_u - u_{n-1}}{h}, \frac{\beta_w - w_{n-1}}{h}) &= 0.
\end{aligned}$$

Keller gives conditions regarding continuity and boundedness of the derivatives of f and g with respect to u, w, u', and w' which ensure that this system has a unique solution [15, pg 96].

The more general boundary conditions of (2.42) can be met by including equations which measure the difference in meeting the boundary conditions, i.e. equations of the form

$$\begin{aligned}
\sigma_r(a) - \alpha_\sigma &= 0, \\
\sigma_r(b) - \beta_\sigma &= 0, \\
w(a) - \alpha_w &= 0, \\
w(b) - \beta_w &= 0.
\end{aligned}$$

Recall that $\sigma_r(a)$ is a nonlinear function of $u(a)$, $w(a)$, $u'(a)$, and $w'(a)$. Similarly, $\sigma_r(b)$ depends on the values of the same quantities at $r = b$. Thus we must also include u_0, w_0, u_{n+1}, and w_{n+1} with the variables for which we must solve. This would give us $2(n+2)$ nonlinear equations in $2(n+2)$ unknowns.

There are many software packages which provide nonlinear equation solvers which can be applied to these systems to arrive at an approximate solution. For example, the FORTRAN subroutines `LMDIF1` and `LMDER1`, part of the MINPACK collection of optimization subprograms, are available from *netlib*. Both subroutines solve nonlinear systems of equations. They differ in their requirement of Jacobian information about the system. MATLAB has a built-in function *fsolve* which also solves a nonlinear system of equations. It can be used with or without providing a Jacobian matrix.

Convergence with the finite difference method compares favorably with the shooting method, although convergence in the problems with traction boundary conditions seems to be much more sensitive to the initial guess. One advantage to this method over the shooting method is that is easier to "see" how changes in parameters may affect the values of u and w at the mesh points. This would be particularly useful in solving a minimization problem where the above functions are constraints. Also, it is possible to formulate the finite difference method without solving the system of equations (3.5). This would avoid problems with singularity of the matrix A. A disadvantage of the finite difference method is that it is only second-order accurate. There are fourth-order finite difference formulas, but they are very awkward to implement. Thus for the finite difference method, higher accuracy is better gained through decreasing the step size, Δr, rather than increasing the accuracy of the method itself.

3.3 Examples of Solutions to the Forward Problem

We present for examples several cases involving different boundary conditions and different functions for the elastic moduli. For all cases, the data was generated by the nonlinear shooting method using a fourth-order method to solve the initial value problems. The plots presented represent cross-sections of the membrane. The entire membrane would be generated by rotating the cross-section about $r = 0$. We assume for all cases that the magnitude of the vertical pressure is $P = 100$, and that the membrane thickness is $h = 1$. Also we specify the interval over which r ranges to be $[0.5, 1.5]$. For simplicity, we assume a flat reference state.

Example 1 Our first case is characterized by the information shown in Table 3.1. Since $E_r = E_\theta$, this membrane is isotropic. We see in Figure 3.1 that the clamped boundary conditions cause a somewhat parabolic deformed state, as we might expect.

Boundary Conditions	$u(a) = 0 \quad u(b) = 0$ $w(a) = 0 \quad w(b) = 0$
Moduli	$E_r(r) = 97$ $E_\theta(r) = 97$ $\nu_{r\theta}(r) = 0.34$

Table 3.1 Functions and boundary conditions characterizing Example 1

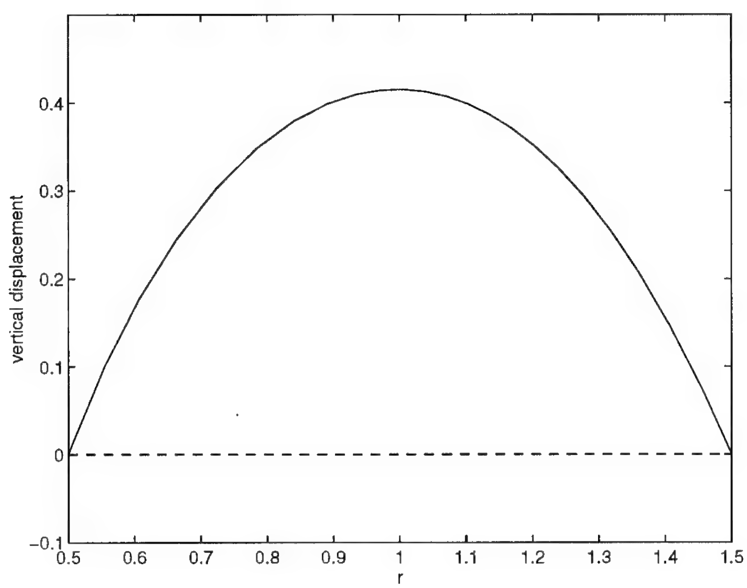

Figure 3.1 Results from solving the forward problem using the information in Table 3.1. The solid line represents the deformed state, while the dashed line represents the flat reference state.

Example 2 Our second example uses different displacement boundary conditions, as well as linear functions for the moduli Again, this membrane is isotropic.

Boundary Conditions	$u(a) = 0 \quad u(b) = 0$ $w(a) = 1 \quad w(b) = 0$
Moduli	$E_r(r) = 50 + 50r$ $E_\theta(r) = 50 + 50r$ $\nu_{r\theta}(r) = 0.25 + 0.1r$

Table 3.2 Functions and boundary conditions characterizing Example 2

Example 3 For the third example, we present an orthotropic membrane where the moduli are constant. We have chosen the values of E_r and E_θ so that the ratio of E_θ to E_r is approximately 1.58. This is compatible with the findings of Gates, McCammond, Zingg, and Kunov [7] in their testing of stiffness properties in the canine diaphragm. Here we also see traction boundary conditions in the form of (2.42). Since u is not required to be zero at the endpoints, we see that the deformed membrane is stretched in the radial direction.

Boundary Conditions	$\sigma_r(a) = 67 \quad \sigma_r(b) = 90$ $w(a) = 1 \quad w(b) = 0$
Moduli	$E_r(r) = 100$ $E_\theta(r) = 158$ $\nu_{r\theta}(r) = 0.25$

Table 3.3 Functions and boundary conditions characterizing Example 3

Example 4 Again, we have an orthotropic membrane in this case, and we have returned to the displacement boundary conditions. However, we now have functions for the Young's moduli which are sinusoidal. The Poisson Ratio remains a constant value. While the difference between this example and example 2 may not be visible in the graphs, we expect that this will be an interesting case for the inverse problem.

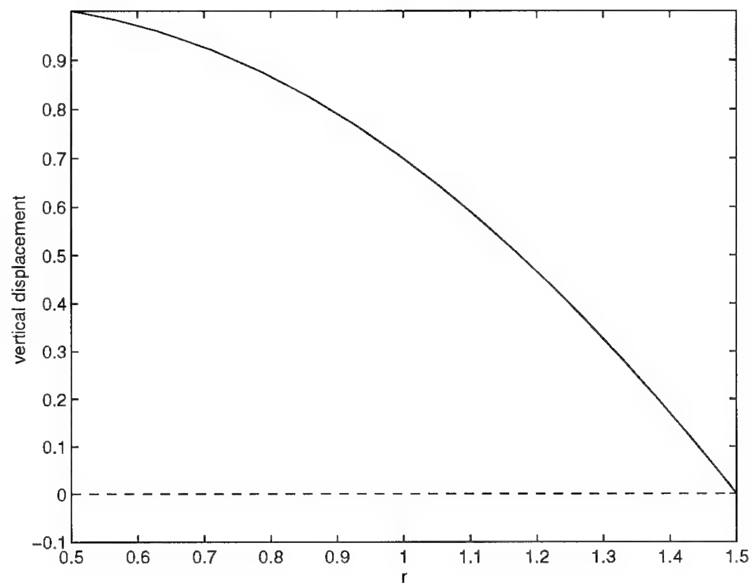

Figure 3.2 Results from solving the forward problem using the information in Table 3.2. The solid line represents the deformed state, while the dashed line represents the flat reference state.

Boundary Conditions	$u(a) = 0 \quad u(b) = 0$ $w(a) = 1 \quad w(b) = 0$
Moduli	$E_r(r) = 95 - 40\cos \pi r$ $E_\theta(r) = 120 - 50\cos \pi r$ $\nu_{r\theta}(r) = 0.25$

Table 3.4 Functions and boundary conditions characterizing Example 4

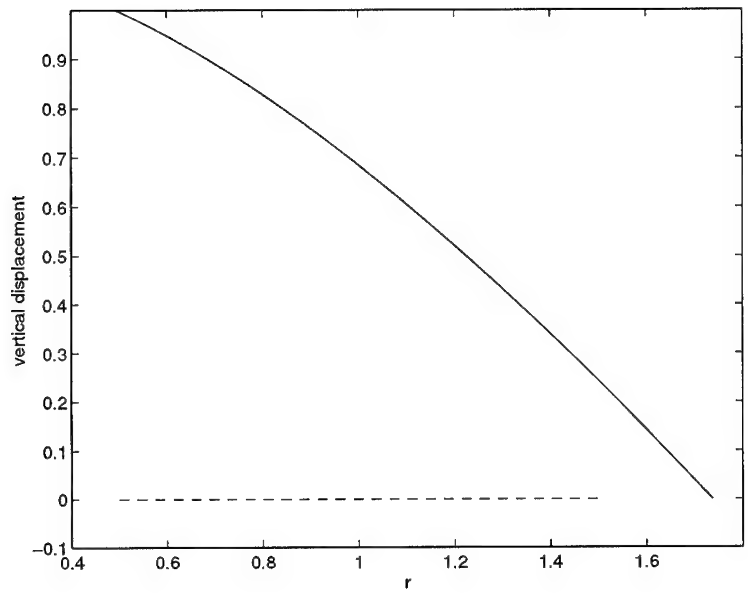

Figure 3.3 Results from solving the forward problem using the information in Table 3.3. The solid line represents the deformed state, while the dashed line represents the flat reference state.

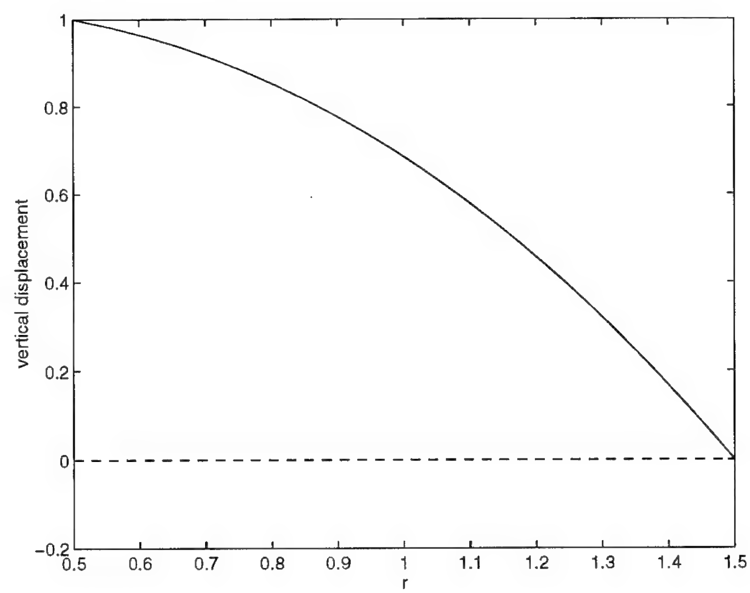

Figure 3.4 Results from solving the forward problem using the information in Table 3.4. The solid line represents the deformed state, while the dashed line represents the flat reference state.

Chapter 4

A Direct Approach to the Inverse Problem

We now turn our interest to the solution of the inverse problem. We assume that we are given observed values of the displacements u and w at several values of $r \in [a, b]$ including the endpoints. This would be equivalent to knowing the values at several points in (a, b) together with the boundary conditions of (2.41). We also know the magnitude of the vertical pressure P, the membrane thickness h, and the reference vertical displacement $z(r)$. The goal of the inverse problem is to find the material properties to account for the observed deformation under the known pressure. In section 4.1 we present a way of solving equations (2.39)–(2.40) in terms of the elastic moduli. We then show examples of how well we can recover these moduli using this approach in section 4.2.

4.1 Solution of the System of Nonlinear Elasticity in Terms of the Moduli

We first consider solving the governing equations of the deformation (2.39)–(2.40) in terms of the elastic moduli themselves. Recall that the moduli occur in (2.39)–(2.40) in the form of the constitutive law that defines σ_r and σ_θ. Thus we first consider a way of solving for these stresses in terms of the displacements. Integrating equation (2.40) from a to r, we find the following relation:

$$\frac{\sigma_r(r+u)(z'+w')}{\sqrt{(1+u')^2+(z'+w')^2}} + \frac{Pr^2}{2h} = C \tag{4.1}$$

where

$$C = \left[\frac{\sigma_r(r+u)(z'+w')}{\sqrt{(1+u')^2+(z'+w')^2}} + \frac{Pr^2}{2h}\right]\Bigg|_{r=a} \tag{4.2}$$

If we assume that we may infer from the observed values of u and w the values of their derivatives, nearly all of the quantities in equation (4.2) are known at $r = a$. The only unknown value is $\sigma_r(a)$ since we do not know the elastic moduli at $r = a$.

The strains on which σ_r depends are known, however. Hence, for given initial values of the elastic moduli, C is a known constant for this problem. However, if we have traction boundary conditions of the form of equation (2.42) or if the value of $\sigma_r(a)$ can be measured experimentally, then C is a known constant regardless of knowledge of the moduli at $r = a$. It is interesting to note that we could have integrated from r to b. This would have changed the value of C to the same quantity evaluated at $r = b$ instead of $r = a$. Then C would be a known constant if the all quantities are known at $r = b$. However, to be concise we will discuss only the case where the quantities are known at $r = a$, and C is specified by equation (4.2).

From equation (4.1) we see that

$$\frac{\sigma_r(r+u)(1+u')}{\sqrt{(1+u')^2+(z'+w')^2}} = \frac{1+u'}{z'+w'}\left(C - \frac{Pr^2}{2h}\right).$$

If $z' + w' \neq 0$, we can substitute this relationship into equation (2.39), which yields

$$\frac{d}{dr}\left\{\frac{1+u'}{z'+w'}\left(C - \frac{Pr^2}{2h}\right)\right\} = \sigma_\theta\sqrt{(1+u')^2+(z'+w')^2} \tag{4.3}$$

Thus equations (4.1) and (4.3) give us the following system of equations.

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \end{pmatrix} = \begin{pmatrix} \frac{d}{\sigma_r(r+u)(z'+w')}\left(C - \frac{Pr^2}{2h}\right) \\ \frac{1}{d}\frac{d}{dr}\left\{\frac{1+u'}{z'+w'}\left(C - \frac{Pr^2}{2h}\right)\right\} \end{pmatrix}. \tag{4.4}$$

Recall that $d = \sqrt{(1+u')^2+(z'+w')^2}$, and let us call the right-hand side of this equation y. Note that, given the assumption that we know the derivatives of u and w, y depends exclusively on known constants and a given value of $\sigma_r(a)$. It should be noted that this assumption of knowing the derivatives is not unreasonable, as the observed data may be easily interpolated by a cubic spline. The resulting interpolant would be twice differentiable, allowing us to construct the right-hand side of this equation. Having arrived at this equation, we must now consider the form of the constitutive law in order to recover the elastic moduli. We will consider the linear isotropic and specially orthotropic cases described in Section 2.2.2.

4.1.1 Linear Isotropic Constitutive Law

For an isotropic material, the linear constitutive law is as shown in equation (2.33). Using this relationship, equation (4.4) becomes

$$\begin{pmatrix} \frac{E}{1-\nu^2}\left(\mathcal{E}_r + \nu\mathcal{E}_\theta\right) \\ \frac{E}{1-\nu^2}\left(\mathcal{E}_\theta + \nu\mathcal{E}_r\right) \end{pmatrix} = y.$$

Thus we have two equations and two unknowns–the Young's Modulus, E, and the Poisson ratio, ν. We expect there to be a unique solution. In order to see this, we introduce the following change of variables.

$$\begin{aligned} \xi_1 &= \frac{E}{1-\nu^2} \\ \xi_2 &= \frac{\nu E}{1-\nu^2} \end{aligned}$$

If $E \neq 0$, we can change variables back using the relationships

$$\begin{aligned} \nu &= \frac{\xi_2}{\xi_1} \\ E &= \xi_1 \left[1 - \left(\frac{\xi_2}{\xi_1} \right)^2 \right] \end{aligned}$$

Since we usually expect $E \gg 0$, this is a valid change of variables. Thus we can write

$$\begin{pmatrix} \mathcal{E}_r & \mathcal{E}_\theta \\ \mathcal{E}_\theta & \mathcal{E}_r \end{pmatrix} x = y \tag{4.5}$$

where

$$x = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Let us call the matrix in equation (4.5) G. Clearly, G is invertible whenever $\mathcal{E}_r^2 \neq \mathcal{E}_\theta^2$. Recalling equations (2.37)–(2.38), we see that this condition is equivalent to

$$|u| \neq r \left| \sqrt{(1+u')^2 + (z'+w')^2} - 1 \right| . \tag{4.6}$$

Note that G depends only on the observed displacements and their derivatives. Hence, if (4.6) holds, the solution vector

$$x = G^{-1} y$$

is determined uniquely for a known value of $\sigma_r(a)$. Thus we can calculate algebraically the functions $E(r)$ and $\nu(r)$ over the entire interval, if we have the necessary values at $r = a$ and if there is no $r \in (a, b]$ such that $w'(r) = -z'(r)$. Even if there is such an r, we may still be able to determine the elastic moduli except in a small neighborhood of this r.

4.1.2 Linear Specially Orthotropic Constitutive Law

The form of the linear specially orthotropic constitutive relationship is given in equation (2.31). Using this equation, we can write equation(4.4) as

$$\begin{pmatrix} \frac{E_r}{1-\nu_{r\theta}\nu_{\theta r}}\left(\mathcal{E}_r + \nu_{\theta r}\mathcal{E}_\theta\right) \\ \frac{E_\theta}{1-\nu_{r\theta}\nu_{\theta r}}\left(\mathcal{E}_\theta + \nu_{r\theta}\mathcal{E}_r\right) \end{pmatrix} = y.$$

Using equation (2.32) resulting from the assumption of hyperelasticity, we may rewrite this equation as

$$\begin{pmatrix} \frac{1}{1-\rho\nu_{r\theta}^2}\left(E_r\mathcal{E}_r + \nu_{r\theta}E_\theta\mathcal{E}_\theta\right) \\ \frac{E_\theta}{1-\rho\nu_{r\theta}^2}\left(\mathcal{E}_\theta + \nu_{r\theta}\mathcal{E}_r\right) \end{pmatrix} = y,$$

where we define

$$\rho = \frac{E_\theta}{E_r}.$$

Thus there are three elastic moduli to recover from this problem. However, since we have only two equations, we can not expect to recover all three. Therefore, let us assume that the major Poisson ratio, $\nu_{r\theta}$ is a known constant. Then we have only the longitudinal and transverse Young's moduli, E_r and E_θ to determine. We therefore introduce the following change of variables:

$$\begin{aligned} \xi_1 &= \frac{E_r}{1-\rho\nu_{r\theta}^2} \\ \xi_2 &= \frac{E_\theta}{1-\rho\nu_{r\theta}^2} \end{aligned}$$

If $E_r \neq 0$, we can change variables back by using the relationships

$$\begin{aligned} \rho &= \frac{\xi_2}{\xi_1} \\ E_r &= \xi_1\left(1-\rho\nu_{r\theta}^2\right) \\ E_\theta &= \xi_2\left(1-\rho\nu_{r\theta}^2\right) \end{aligned}$$

Again, we expect that $E_r \gg 0$. Thus we consider this a valid change of variables. Using these new variables, we can write

$$Gx = y \tag{4.7}$$

where

$$G = \begin{pmatrix} \mathcal{E}_r & \nu_{r\theta}\mathcal{E}_\theta \\ 0 & \nu_{r\theta}\mathcal{E}_r + \mathcal{E}_\theta \end{pmatrix}.$$

This matrix is singular when $\mathcal{E}_r = 0$ or when $\nu_{r\theta}\mathcal{E}_r = -\mathcal{E}_\theta$. Otherwise, we have a solution vector

$$x = G^{-1}y$$

which is uniquely determined for a given value of $\sigma_r(a)$, and the constant $\nu_{r\theta}$. Thus if there is no $r \in (a, b]$ such that $w'(r) = -z'(r)$, we can calculate algebraically the functions $E_r(r)$ and $E_\theta(r)$. Once these functions are recovered, we would like to use equations (2.34)–(2.35) to recover the moduli of the matrix and fiber. If the fiber and matrix are both isotropic and the volume fractions v_f and v_m are known, these equations form a nonlinear system of equations in two unknowns which we should be able to solve for E_f and E_m.

4.2 Examples of Recovered Moduli

We present as examples results from recovering the moduli from the four examples in section 3.3. In every case, we use the data for u and w with their first and second derivatives which was generated by the forward solution of the problem using the shooting method. We assume that $\sigma_r(a)$ is known for all cases. In example 3, this simply involves the boundary conditions. For other cases, we calculate this value using known initial values of the elastic moduli.

Example 1 The parameters of this problem were specified in Table 3.1. Since this is an isotropic problem, we expect to recover both the Young's modulus and the Poisson ratio. We can see in Figure 3.1 that there is a critical point where $w' = 0$. Thus we expect to lose accuracy in recovering the moduli near this critical point. Figure 4.1 shows the results of this direct method using $\sigma_r(a) = 78.0146$. We clearly see a spike in both moduli near this critical point. However, the effect seems to be localized, since the recovered values seem to be very close to the true values except for only a few points. Thus, except for a small neighborhood near the critical point we are able to recover the moduli over the entire interval with good accuracy.

Example 2 In Figure 3.2 we can see that we should not have the same difficulty with critical points. So we hope to find the correct values for E and ν at all points on the interval. Using $\sigma_r(a) = 44.5845$ we get the results shown in Figure 4.2 Clearly, all recovered values are very close to the true values.

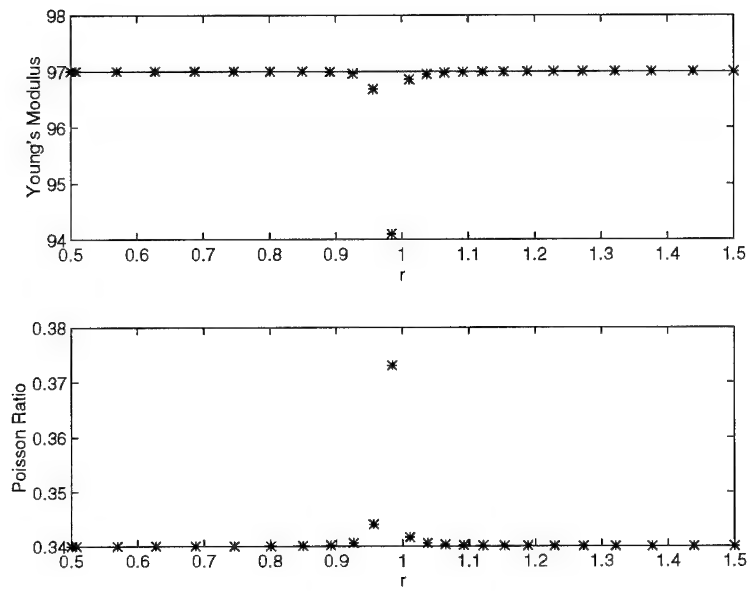

Figure 4.1 Results from recovering the elastic moduli for Example 1. The solid line represents the true values of the moduli while the stars represent the recovered values of the moduli.

Example 3 Since this problem is an orthotropic membrane, we can only recover E_r and E_θ. We assume that the value of $\nu_{r\theta} = 0.25$ is known. For this example, we use the value of $\sigma_r(a)$ provided by the boundary conditions shown in Table 3.3. Again, we see very good results in recovering the true moduli.

Example 4 For this orthotropic problem, we again assume that the value of $\nu_{r\theta} = 0.25$ is known. Recall that the functions describing the moduli as shown in Table 3.4 are sinusoidal. Figure 4.4 shows the results of recovering these moduli using $\sigma_r(a) = 49.6505$. There appears to be slightly more error in recovering these moduli than in the other examples. This may be due to the wider variation in moduli values than has been present in the other examples

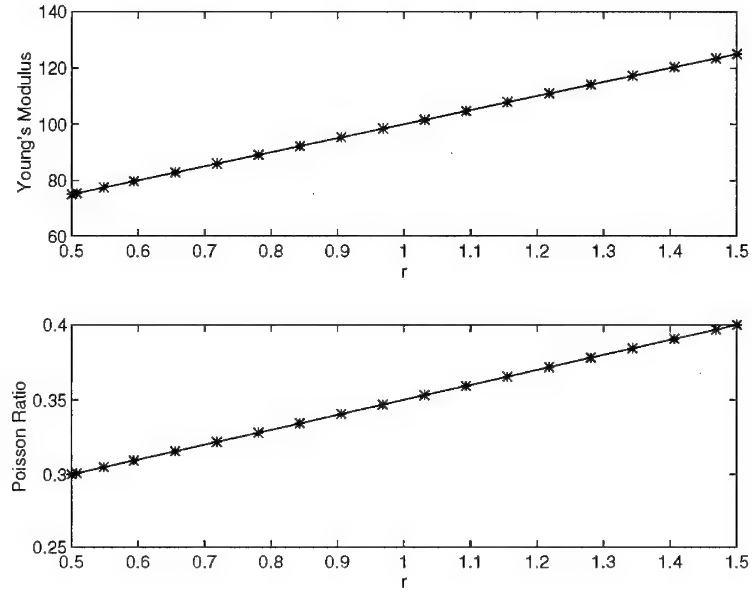

Figure 4.2 Results from recovering the elastic moduli for Example 2. The solid line represents the true values of the moduli while the stars represent the recovered values of the moduli.

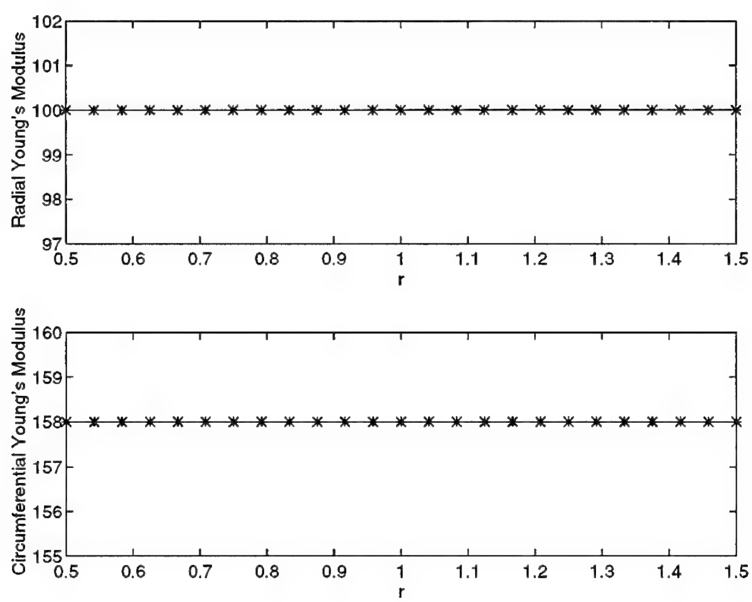

Figure 4.3 Results from recovering the elastic moduli for Example 3. The solid line represents the true values of the moduli while the stars represent the recovered values of the moduli.

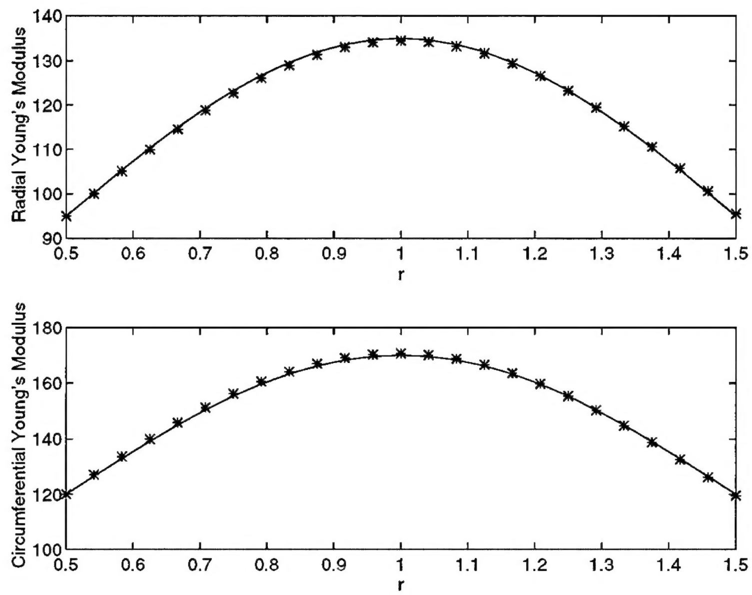

Figure 4.4 Results from recovering the elastic moduli for Example 4. The solid line represents the true values of the moduli while the stars represent the recovered values of the moduli.

Chapter 5

Concluding Remarks

In this thesis, we have focused our attention on recovering the elastic moduli of axisymmetric, nonlinearly elastic membranes which follow linear constitutive laws. We have shown that with knowledge of the deformation at the boundaries and on the interior, we can determine the elastic moduli quite accurately, if we also know the radial stress at either boundary. Practical application of this method of recovering the moduli has the following steps:

1. Obtain measurements of vertical and radial displacement of the deformed membrane, as well as measurements of the vertical pressure, membrane thickness, and the radial stress at the inner radius.

2. Interpolate the measurements of displacement with a cubic spline or other interpolant which has at least two continuous derivatives.

3. At each point where the moduli are to be determined, solve equation (4.4) for the moduli using the interpolants to obtain the necessary derivatives. This solution may involve a change of variables to make (4.4) a linear system.

Thus the determination of these parameters comes from direct solution of algebraic equations. A possible area for additional research with this method would be in solving an optimization problem for the correct value of radial stress at the boundary. The solution to such a problem might eliminate the necessity for knowing this value prior to determining the moduli.

We have also presented a model for star-like membrane deformation which is much more general than the axisymmetric case. This model is general enough to be applied to the diaphragm with far fewer simplifying assumptions. It is hoped that further research using this model will allow more accurate estimation of the parameters describing deformation of the diaphragm.

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